GEOMETRIC PROBABILITY APPLICATIONS THROUGH HISTORICAL EXCURSION

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ABSTRACT

The aim of the contribution is to show how the discussion of the history of geometric probability can reveal its simplest application possibilities and its numerous interdisciplinary relations, and how it can attract a deserved interest of students in this fascinating branch of science. The paper summarizes the workshop held by the author at the conference ESU-6; materials provided to participants for the experiments are available at the Internet address "http://euler.fd.cvut.cz/∼hyksova/probability".

1 Introduction

*For most students and even for many teachers, probability theory belongs to the least favorite parts of school mathematics, although it relates to our lives more than anything else; in fact, we are surrounded by randomness and constantly face situations where real outcomes are not completely certain, so that probability estimations come into play. The key problem is perhaps the fact that school probability calculus is usually restricted to tossing coins and throwing dices, i.e., to problems that do not a bit seem to relate our everyday lives and to illustrate the importance and the wide applicability of probability theory. The aim of this contribution is to show how the theory of geometric probability – in addition to other probabilistic topics – can help to persuade students that probability theory plays a key role in many fields closely related to our health, safety, wealth etc.

Similarly as the standard probability, the theory of geometric probability originated as a tool of spatial games understanding and for a detailed elaboration of the rules ensuring their fairness. Later on, it turned out to be a substantial tool for extracting quantitative information on spatial objects (e.g., the percentage composition of a rock, the average volume of the crystalline mineral grains, the area of grain boundary surfaces or the length of grain edges per unit volume of a rock etc.; similarly it is used to get information on cells in a tissue, tumors or lesions in organs, vessels etc.) from probes of a lower dimension (sections or microscope images, linear or point probes). Recall that when we pass from the investigation of properties of finite populations to geometric properties (e.g., volume, area, length, shape) of geometrically describable objects (e.g., human beings and their organs and blood vessels, animals, plants, cells in a tissue, rivers, rocks etc.), replacing random population samples by probes of a lower dimension mentioned above, and instead of "classical" probability we base our inferences on geometric probability, we pass from statistics to the domain of *stereology*. Since we are not only surrounded but even formed by such structures, stereology and

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hence also geometric probability are of great importance to our lives and represent a substantial tool for exploring and understanding the world around us.

Mathematically, geometric probability was introduced as the extension of the classical definition of probability to situations with uncountable number of cases: instead of the number of favourable and all possible cases it is now necessary to work with measures of relevant sets. For example, probability that a point X lying in a set C lies also in a subset $B \subseteq C$ is defined as the ratio

$$P(X \uparrow B | X \uparrow C) = \frac{m(B)}{m(C)}, \tag{1}$$

where m denotes a convenient measure of the given set (e.g., area or volume for a set in 2D or 3D, respectively; see also footnote 3). In the following sections we will see that from this simple definition, useful applications immediately follow.

2 Points in plane – area estimation

The first known discussion of a problem concerning geometric probability appeared in the private manuscript of Isaac Newton (1643 – 1727) written between 1664 and 1666 (Newton, 1967): consider a ball of negligible size falling perpendicularly upon the centre of a horizontal circle divided into two unequal sectors B_1, B_2 (Fig. 1 left); suppose that the ratio of areas of these sectors is $2 : \sqrt{5}$ and the prospective win for a player is equal to b_1 in the case that the ball falls into the sector B_1, and b_2 if it falls into B_2. Newton claims that the "hopes" of the player worth

$$\frac{2b_1 + \sqrt{5}\, b_2}{2 + \sqrt{5}}.$$

His aim was to show that the ratio of chances can be irrational; nevertheless, from today point of view we can say that he provided the first area fraction estimation: let us set $b_1 = 1$, $b_2 = 0$ (i.e., the player wins 1 unit if the ball hits the sector B_1, nothing otherwise); the hopes $2/(2 + \sqrt{5})$ express directly the probability of hitting B_1. On one hand, it is equal to the ratio of the area of the sector B_1 and the whole circle $C = B_1 \cup B_2$; on the other hand, the same probability can be expressed as the ratio of the mean number N_{hits}, how many times the ball hits the sector B_1, provided it is thrown N_{total} times on the circle C, and the total number of throws N_{total} :

$$P(X \uparrow B_1 \,|\, X \uparrow C) = \frac{\overline{N_{hits}}}{N_{total}} = \frac{A(B_1)}{A(C)}. \tag{2}$$

If the area $A(C)$ is known, the equation (2) can be used for the estimation of the area $A(B_1)$ by counting the number of throws and hits. Denoting the average fraction of hits by the symbol P_P, and the area fraction by A_A, we immediately obtain one of the fundamental formulas of stereology:

$$P_P = A_A. \tag{3}$$

Instead of "throwing" isolated random points, it is possible to place a point grid randomly on C and calculate the ratio of points hitting any set of interest $B \subseteq C$ and

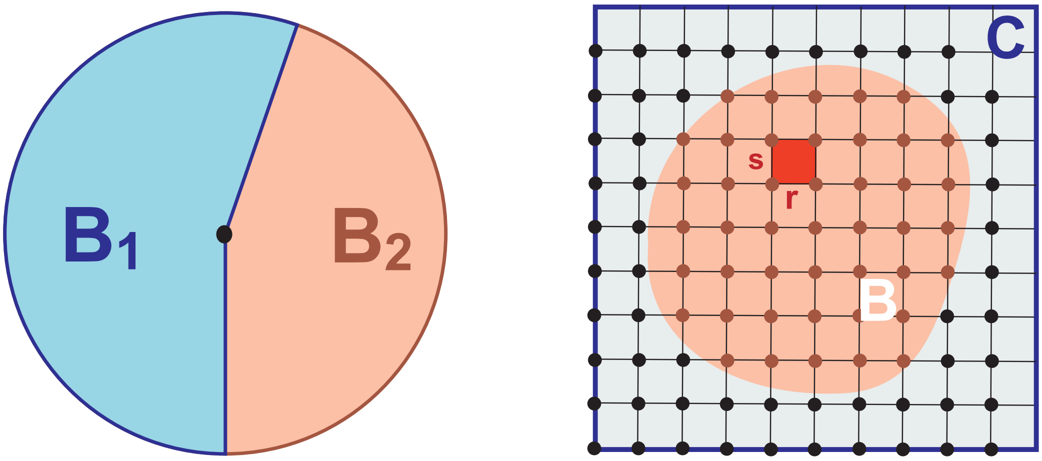

Figure 1: Circle sectors from Newton's example (left), planar point grid (right)

C. Finally, considering C to be the rectangle of the area $rs \cdot N_{total}$ formed by the grid (see Fig. 1 right), we obtain the direct estimation of the absolute size of the area $A(B)$:

$$A(B) = rs \cdot \overline{N_{hits}} \,. \tag{4}$$

Figure 2: Simple classroom application of the point-counting method

In the classroom, it is possible to test the described "point-counting method" in simple experiments and to compare it with some other possibilities how to estimate an unknown area of some complicated region, that can be suggested by students themselves (e.g., using a millimeter graph-paper). It is sufficient to print a point grid on a plastic sheet and place it randomly on the investigated area – for example, a map of a country, lake, group of lakes or islands etc. Several such materials were presented in the workshop and are available at the above mentioned Internet address.

Figure 2 illustrates the estimation of total area of islands named Franz Josef Land (their official land area is 16 134 km^2) by the point-counting method.[2]

3 Points in space – volume estimation

Similar ideas can be repeated in 3D. Probability that a randomly selected point X that hits a set C hits also a subset $B \subseteq C$ was introduced as the ratio of volumes of these sets:[3]

$$P(X \uparrow B \,|\, X \uparrow C) = \frac{\overline{N_{hits}}}{N_{total}} = \frac{V(B)}{V(C)}\,. \tag{5}$$

Consider C to be a prism with the volume $V(C) = rst \cdot N_{total}$ formed by the spatial point grid depicted in Fig. 3 (left) and randomly placed on the set B. Then the volume $V(B)$ can be estimated by

$$V(B) = rst \cdot \overline{N_{hits}}\,. \tag{6}$$

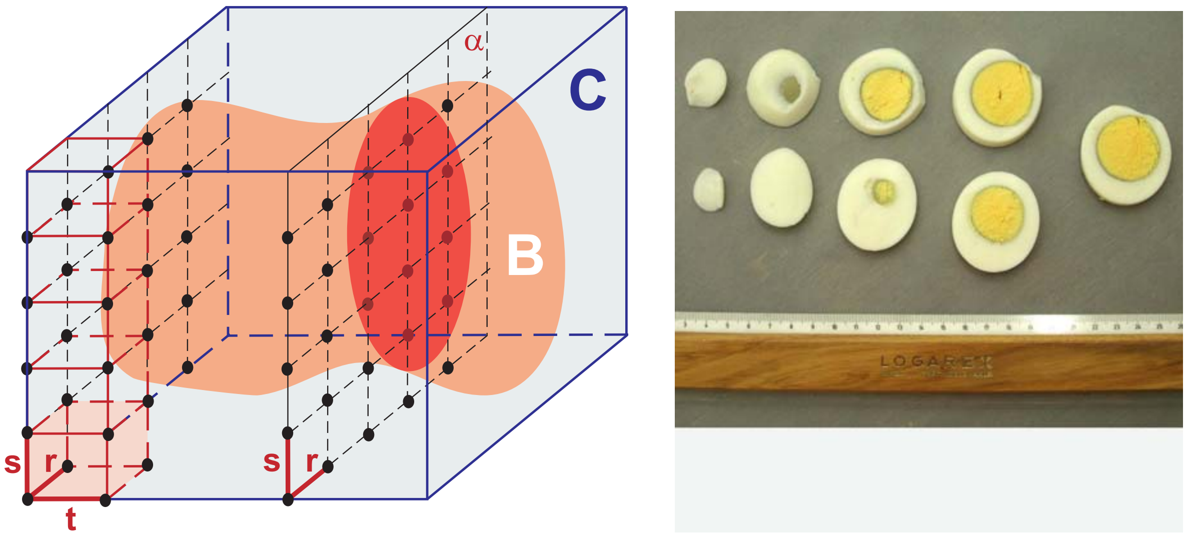

Figure 3: Spatial point grid (left), volume estimation of an egg (right)

In geology (but also in metallurgy or biomedicine) we often search the volume fraction V_V of some phase in the sample. If the phase is homogeneously distributed in the sample, the restriction of the points considered in equation (5) to any of parallel planes

[2]From Fig. 2, we obtain an estimation 16,250 km^2. For accuracy of the point-counting procedure, one of the simplest formulas gives the following total number of lattice points P_T necessary to achieve the desired variance: $P_T = P_P(1-P_P)/\sigma^2(P_P)$. For more details see e.g. (Underwood, 1970).

[3]Hadwiger theorem proved in (Hadwiger, 1957) implies that the space of "convenient measures" in the formula (1), i.e., measures that are invariant under the group of Euclidean motions, is of dimension $n+1$ in an Euclidean space of dimension n, and volume is only one of them. An accessible exposition can be found in (Klain and Rota, 1997).

formed by the points of the spatial grid (e.g., plane α in Fig. 3 left) gives the same average fraction as the whole grid. That is, the average areal fraction A_A determined on this section is equal to the average volume fraction V_V of the investigated phase in the sample and it is equal to the average fraction of hitting points:[4]

$$P_P = V_V = A_A. \tag{7}$$

The idea of determination of the volume of some object or volume fraction of its specific phase from areal estimations in sections by systematically random parallel planes[5] can easily be illustrated by the estimation of the volume of an egg. Sections by systematically random system of parallel planes are represented by slices made by an egg slicer, into which an egg is randomly placed (see Fig. 3 right). Planar point grids in sections form together a spatial point grid. Volume of an egg can thus be estimated using equation (6), where N_{hits} is calculated in individual selections and added together. The result can be compared with another volume measurement – e.g., by placing the egg into a graduated cylinder with water. Similarly, the volume fraction of a yolk in the egg can be estimated from the estimation of its areal fraction in particular slices. As it was mentioned above, in the case of the spatial homogeneous distribution of the investigated phase, the volume fraction estimation can be based on one or only several sections as a random sample, using the equation (7). It should be stressed that an egg is used only for a better understanding the core of the method, which is especially effective for dealing with finely dispersed phases such as small particles in a rock or a metal, cells in a tissue etc. Formerly the grid was inserted in the eyepiece of the microscope or superimposed on a photomicrograph; in the age of computers, estimations can be done using a convenient image processing software.

At the beginning of the previous section, the example formulated by Isaac Newton was discussed. Another – but far more famous and influential – example is connected with the name of Georges-Louis Leclerc, later Comte de Buffon (1707 – 1788), and it will be discussed in the next section. The systematic development of geometric probability started with the treatises published by Morgan William Crofton (1826 – 1915) who derived the key theorems concerning straight lines in a plane and briefly outlined possible generalization to lines in space (Crofton, 1868 and 1885); this generalization was done in full details by Emanuel Czuber (1851 – 1925), who published the first monograph entirely devoted to geometric probability (Czuber, 1884). Czuber payed a great attention to the definition of geometric probability as a content ratio in $\mathbb{R}^n$; in the special case of $\mathbb{R}^2$ and $\mathbb{R}^3$ in the sense of (2) and (5).[6]

It is rather surprising that even though the above mentioned point-counting method for volume estimation follows straightforward from the definition available in the 1880's, it took half a century before it was discovered by geologists. Let us remark that long before geometric probability was systematically developed, Achille Ernest Oscar Joseph Delesse (1817 – 1881) introduced a mechanical method for the estimation of the volume density of a mineral homogeneously distributed in a rock, based on the measurement

[4]For an exact proof see e.g. (Underwood, 1970) or (Baddeley and Vedel Jensen, 2005).

[5]The first plane is placed randomly into the investigated object and the others are parallel with the constant distance.

[6]For more details on the history of geometric probability see e.g. (Czuber, 1899), (Kalousová, 2008), (Saxl and Hykšová, 2009), (Seneta at al., 2001) and (Todhunter, 1865).

of the area density of the investigated mineral in a random plane section (Delesse, 1847 and 1848). The idea that 3D reconstruction of an object is not necessary for its volume estimation, was indeed revolutionary. Nevertheless, practical implementation of the method was still rather laborious: Delesse proposed to cover the polished, enough large random section with an oiled (and thus transparent) paper, trace exposed portions of particular minerals, using different colors for different phases, glue the paper on a tin foil, cut the mosaic and weight particular groups of the same color; areal fractions of minerals correspond (for a homogeneous foil of constant width) to weight fractions, and as mentioned above, also to desired average volume fractions.

Practical methods for quantitative analysis of rocks continued to develop independently of the theory of geometric probability. From various modifications of Delesse's method, let us mention the contribution of August Rosiwal (1860 – 1923), published half a century later (Rosiwal, 1898). Inspired by Delesse, Rosiwal continued in simplification even further and showed that instead of measuring areas of planar regions, it was sufficient to measure lengths of line segments on a line system superposed with the section. In other words, he proved that the average volume fraction of a mineral is equal to its average area fraction and it is equal to its average linear fraction:

$$V_V = A_A = L_L \,. \tag{8}$$

As it was already mentioned, it took more than another 30 years before the point-counting method was discovered and started to be used by geologists: it was described by Andrej Aleksandrovich Glagolev (1894 – 1969) who proved what can be in today terminology expressed by the formula (see (Glagolev, 1933)):

$$V_V = A_A = P_P = L_L \,. \tag{9}$$

4 Curves in plane – length estimation

Perhaps the most famous example from the field of geometric probability (and often also the only one that secondary school students meet) is Buffon's needle problem presented for the first time at the session of French Académie Royal des Sciences in 1733 (see the report in (Fontenelle, 1735)) and published in full details in (Buffon, 1777). Recall the formulation of the problem: *A slender rod is thrown at random down on a large plane area ruled with equidistant parallel straight lines; one of the players bets that the rod will not cross any of the lines while the other bets that the rod will cross several of them. The odds for these two players are required.* In other words, we search the probability that the rod hits some line.

Denote by d the distance between parallels and ℓ the length of the rod, $\ell < d$. Buffon solved this problem with the help of integral calculus and showed that the hitting probability equals $2\ell/\pi d$. Barbier (1860) provided the solution without the use of integral calculus, which can be loosely described in the following way (for more details, see (Kalousová, 2009) and (Klain and Rota, 1997)). Let N be the number of intersections of a randomly dropped needle of length ℓ with any of the parallel straight lines. If $\ell < d$, it can take only the values 0 and 1, otherwise it can take several integer values. Let p_n denote the probability that the needle hits exactly n parallels, and $E(N)$ the mean number of intersections (or the expectation of the random variable N). We can write

$$E(N) = \sum_{n \geq 0} n p_n \,. \tag{10}$$

For $\ell < d$ we have $E(N) = 0 \cdot p_0 + 1 \cdot p_1 = p_1$, which is the probability we search.

Now consider two needles of lengths ℓ_1, ℓ_2, let N_1, N_2 denote the numbers of their intersections with given parallels. Provided the needles are not bound together, the random variables N_1 and N_2 are independent and the mean value of the total number of intersections can be written as the sum

$$E(N_1 + N_2) = E(N_1) + E(N_2)\,. \tag{11}$$

If the needles are welded together at one of their endpoints, N_1 and N_2 are no more independent; nevertheless, the expectation remains additive and the equation (11) holds again. Similarly for m needles bound together to form a polygonal line of arbitrary shape: $E(N_1 + N_2 + \cdots + N_m) = E(N_1) + E(N_2) + \cdots + E(N_m)$.

Since the mean number of intersections $E(N_i)$ evidently depends on the needle length ℓ_i, we can write $E(N_i) = f(\ell_i)$. Equation (11) gives

$$f(\ell_1 + \ell_2) = f(\ell_1) + f(\ell_2) \tag{12}$$

for any $\ell_1, \ell_2 \in \mathbb{R}$. The function $f(\ell)$ is thus linear,[7] i.e., there exists a constant $r \in \mathbb{R}$ such that

$$f(\ell) = r\ell \quad \text{for all } \ell \in \mathbb{R}\,. \tag{13}$$

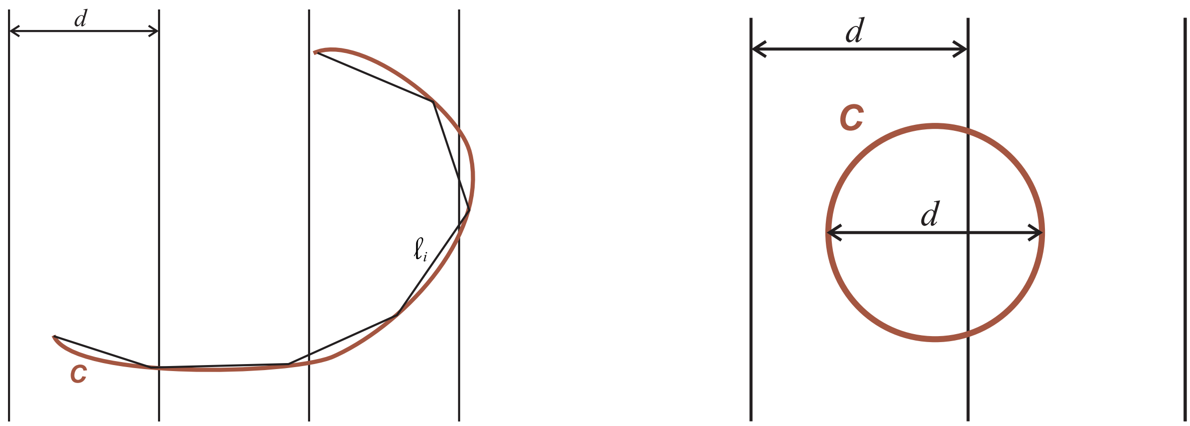

Figure 4: Buffon's needle problem

Now any planar curve C of the length ℓ can be approximated by polygons (Fig. 4 left); passing to the limit, we obtain the mean number of intersections to be again $E(X) = r\ell$. Thus, to determine the value of the constant r, it is sufficient to consider one curve of a convenient shape. Let us simply choose a circle of diameter d (Fig. 4 right) which intersects the system of parallels always in two points, so that $2 = r\pi d$, i.e., $r = 2/(\pi d)$. For the curve of any length ℓ we thus have

$$E(N) = \frac{2\ell}{\pi d} \tag{14}$$

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[7]Equation (13) implies $f(0) = f(0+0) = f(0)+f(0)$, so that $f(0) = 0$; $f(2) = f(1+1) = f(1)+f(1)$, similarly for any natural s, t : $f(s) = s \cdot f(1)$, $f(1) = t \cdot f(1/t)$, thus $f(s/t) = f(s)/f(t)$. In other words, f restricted to rational values of ℓ is linear. Since f is a monotonically increasing function with respect to ℓ, we infer that the equation (13) holds for all $\ell \in \mathbb{R}$.

and for a needle with the length $\ell < d$ this equation gives directly the searched hitting probability:

$$E(N) = p_1 = \frac{2\ell}{\pi d}\,. \tag{15}$$

In the school mathematics, the only "application" of Buffon's needle problem is usually the estimation of the number π. Nevertheless, even a non-excellent student can realize that more effective and more exact methods exist for this aim. What is much more important, is another application: in many cases, it is too complicated to measure exactly the length of some curve or a system of curves (e.g., a vascular bundle, a watercourse, cell boundaries in a tissue section etc.); we can "throw" a system of parallels on its picture, calculate the number of intersections N and use (14) for the estimation of the unknown length:

$$[\ell] = \frac{\pi d}{2} \cdot N\,. \tag{16}$$

In today terminology, the considered system of parallels represents a test system of lines with the length intensity (the mean length in a unit area) $L_A = 1/d$ (see Fig. 5 left). Using this notation, we can write (16) in the form:

$$[\ell] = \frac{\pi}{2} \cdot \frac{1}{L_A} \cdot N\,. \tag{17}$$

Dividing both sides by the area A of the region covering the investigated curve, and denoting by $N_L = N/(AL_A)$ the number of intersections for a unit length of the test system, we obtain one of the fundamental formulas of stereology, which provides an unbiased estimator of the length intensity $\ell_A = \ell/A$:

$$[\ell_A] = \frac{\pi}{2} \cdot N_L\,. \tag{18}$$

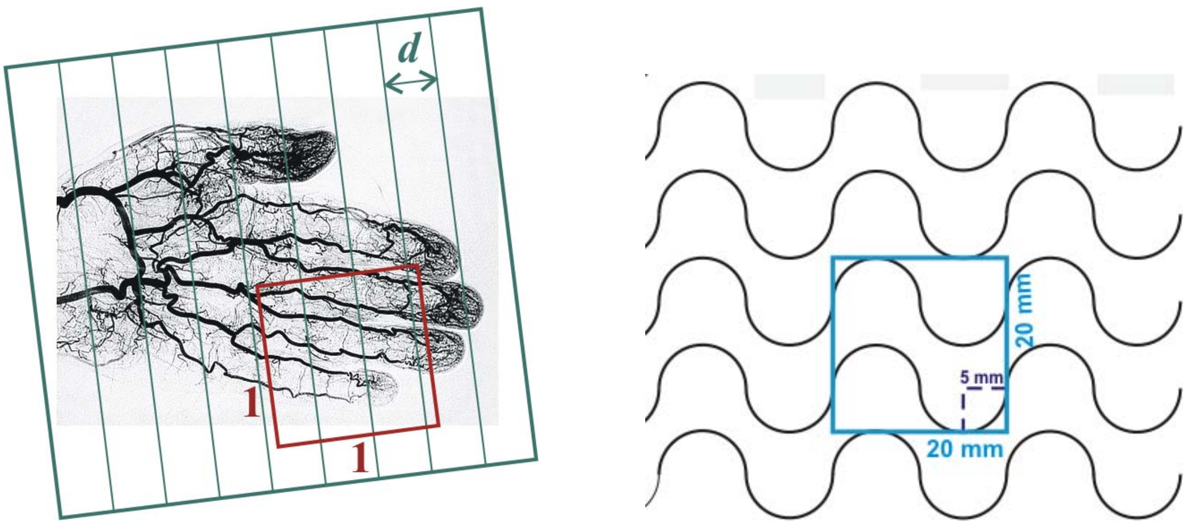

Figure 5: Length estimation based on the needle problem

The equation (18) holds even in the case that the test system is formed not by parallels, but by a system of curves (Fig. 5 right) or a unique curve of constant length intensity.

For students, length estimation is a much more convincing and real application of the needle problem, and they can try to implement it themselves, using for example a map of a river or a highway, as shown in Fig. 6. They can use plastic sheets with parallels and with the system of isotropic curves (Fig. 5 right) and discuss which one is more convenient in this case and why.

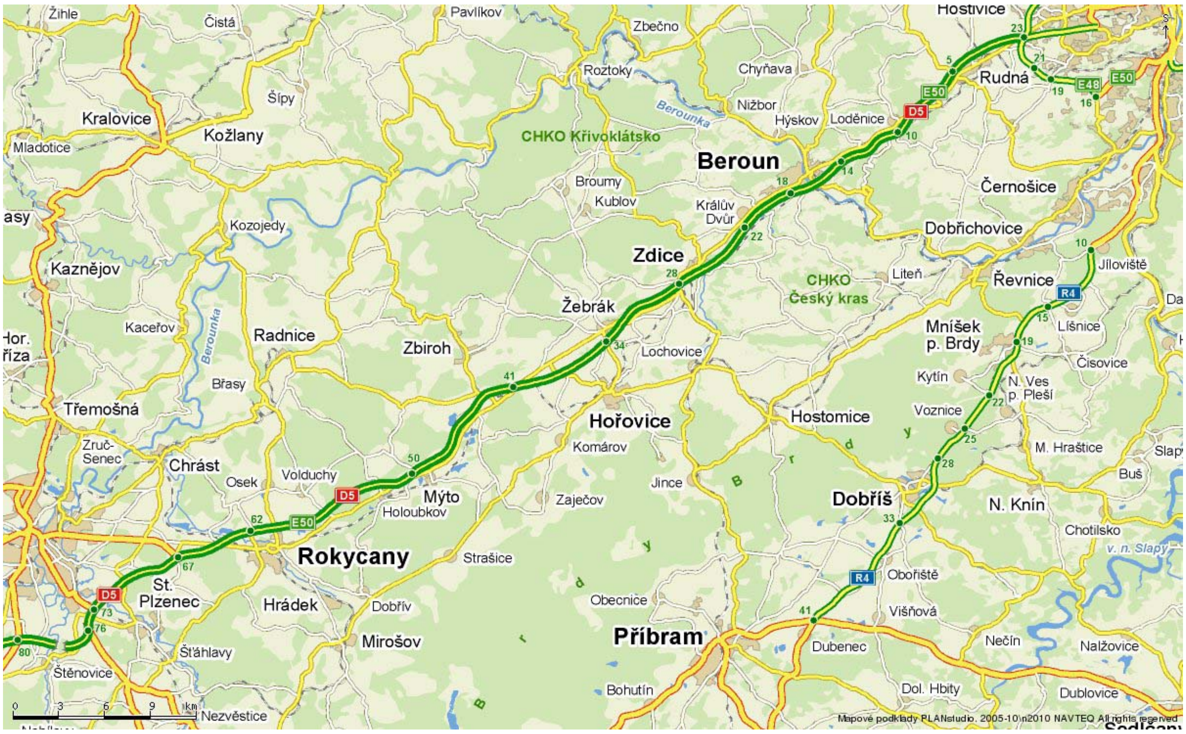

Figure 6: Estimation of the length of River Berounka

After its publication, Buffon's problem remained unnoticed till the beginning of the 19^{th} century. Without a reference to its author, the needle problem was mentioned by Pierre-Simon de Laplace (1749 – 1827), who proposed to use it for estimating the length of curves and area of surfaces; nevertheless, he gave only one example, namely the circumference of a unit circle (Laplace, 1812). Since then, other mathematicians introduced some generalizations; for example, Isaac Todhunter (1820 – 1884) considered a rod of the length rd, a cube and an ellipse (Todhunter, 1857), later also a closed curve without singular points (Todhunter, 1862). Gabriel Lamé (1795 – 1870) included Buffon's needle problem and its generalizations to a circle, an ellipse and regular polygons in his lectures held at École normale supérieure. Inspired by these lectures, Joseph-Émile Barbier (1839 – 1889) published the general theorem concerning the mean number of intersections of an arbitrary curve with the system of parallels (14), and what is indeed remarkable, he replaced equidistant parallels by an arbitrary system of lines and even by a unique curve of constant length intensity (in his words, the total length of the line system in each square metre of the plain was the same) and came to the estimator (18). And what is even more remarkable, he continued in generalizing further to 3D and formulated three more theorems that express other today fundamental stereological formulas used for the estimation of surface area or curve length; for more details and

Barbier's original formulations see (Kalousová, 2009).

5 Lines in plane – size estimation

Let us finally mention Crofton-Cauchy formula explicitly derived in (Crofton, 1868): *The mean breadth of any convex area is equal to the diameter of a circle whose circumference equals the length of the boundary.* In other words: let b denote the breadth of a convex figure X in $\mathbb{R}^2$, that is, the projection of X into the normal to the given direction (see Fig. 7 left), and let L denote its perimeter; using Cauchy theorem, Crofton proved the following formula for the mean projection length of X into the isotropic bundle of directions:

$$w = \frac{1}{\pi}\int_0^\pi b d\varphi = \frac{L}{\pi}, \qquad \text{thus} \qquad L = \pi w\,. \tag{19}$$

Notice that for a circle, the breadth in any direction is $2r$, so that (19) gives the usual formula for the circle perimeter $L = 2\pi r$. What is remarkable is the fact that also for any other convex figure, the perimeter can be simply computed by multiplying the mean breadth by the number π. On the other hand, we can calculate for example the mean breadth of a square with the side a, which is $w = 4a/\pi$.

Now consider a system of parallels with $d > w$ "thrown" on X. The hitting probability is equal to $L/(\pi d)$ [8] and using (19), we can write $p = w/d$. Repeated throwing of test lines on the given figure and counting the fraction of successful hits thus provides the estimation of the mean breadth w. For example, students can try to check whether the Emmenthal cheese indeed contains the eyes of a proper size (Fig. 7 right).

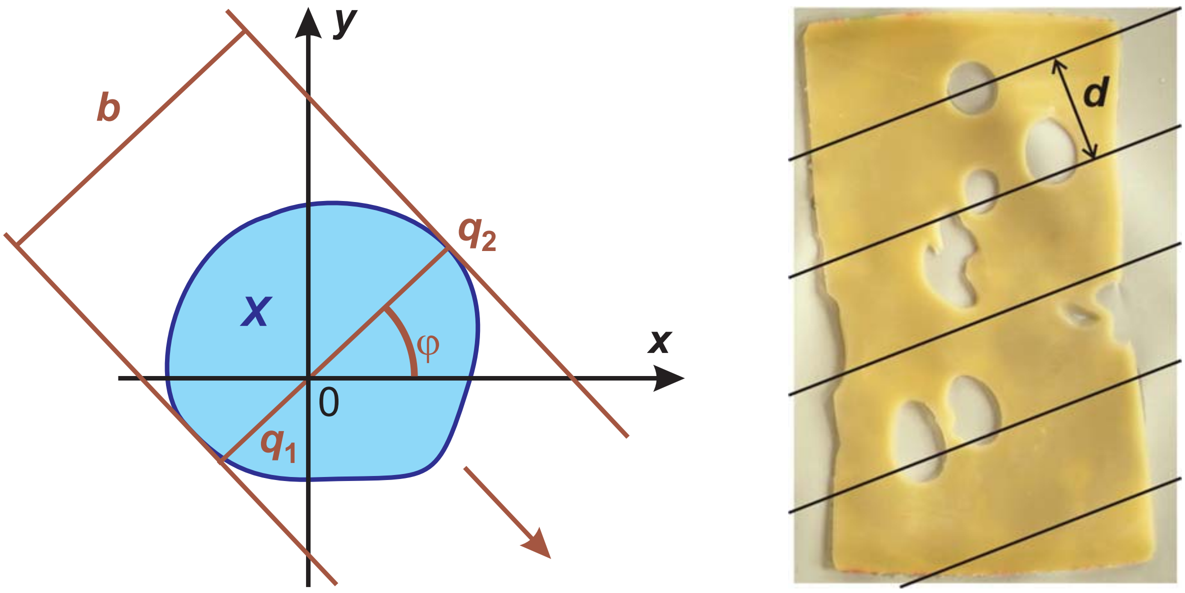

Figure 7: Mean breadth estimation

[8] For a circle with the radius $r < d/2$, the hitting probability equals

$$\frac{2r}{d} = \frac{2\pi r}{\pi d} = \frac{L}{\pi d}\,.$$

Using (14), we can derive the same formula for any convex figure with the perimeter L.

Let us finally remark that Czuber (1884) proved the spatial analogy of Crofton-Cauchy formula and proved that for any convex body, the quadruple of the mean projection area is equal to the surface of the body. Thus, for a sphere we have the usual formula $S = 4\pi r^2$; for a cube we can determine the mean projection area to be $6a^2/4$.

6 Conclusion

The aim of the paper was to awaken an interest in the part of probability theory that is usually neglected in the school mathematics, although its importance and the number of applications has substantially increased since the beginning of the 20$^{\text{th}}$ century when examples from this field were even included among exercises for the leaving examination. Geometric probability can help teachers to answer convincingly the usual students' question (whether in the connection with probability or mathematics in general): "What is it for?" Here they can observe important applications in biomedicine (dermatology, nephrology, oncology, cardiology etc.), material engineering (quantitative analysis of metals, composites, concrete, ceramic), geology etc.

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