THE CONSTRUCTION, BY EUCLID, OF THE REGULAR PENTAGON

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ABSTRACT

We present a modern account of Ptolemy's construction of the regular pentagon, as found in a well-known book on the history of ancient mathematics (Aaboe [1]), and discuss how anachronistic it is from a historical point of view. We then carefully present Euclid's original construction of the regular pentagon, which shows the power of the method of equivalence of areas. We also propose how to use the ideas of this paper in several contexts.

Key-words: Regular pentagon, regular constructible polygons, history of Greek mathematics, equivalence of areas in Greek mathematics.

1 Introduction

This paper presents Euclid's construction of the regular pentagon, a highlight of the *Elements*, comparing it with the widely known construction of Ptolemy, as presented by Aaboe [1]. This gives rise to a discussion on how to view Greek mathematics and shows the care on must have when adopting adapting ancient mathematics to modern styles of presentation, in order to preserve not only content but the very way ancient mathematicians thought and viewed mathematics. [1]

The material here presented can be used for several purposes. First of all, in courses for prospective teachers interested in using historical sources in their classrooms. In several places, for example Brazil, the history of mathematics is becoming commonplace in the curricula of courses for prospective teachers, and so one needs materials that will awaken awareness of the need to approach ancient mathematics as much as possible in its own terms, and not in some pasteurized downgraded versions. [2] As a matter of fact, this text has been used in a course for future secondary school mathematics teachers.

Secondly, it can also be used in secondary education. In many countries, most secondary education mathematics textbooks present just a series of results, some of them justified by examples, the use of analogies and heuristics, but do not show a single proof. So, using Frank Lester's words, many, many, students finish their schooling without ever "meeting the queen". [3] The construction of the pentagon is genuine mathematics, dealing with a result that arouses interest and at the level of secondary school students and it very suited to show the power of the deductive method in mathematics.

In the third place, it is important to show students or readers how mathematics changed along the centuries, how its tools and techniques evolved, and how to appreciate the mathematical accomplishments of past generations, what kind of problems they attacked, how their results were communicated. Discussions on these subjects would certainly enlarge the cultural awareness of students and readers.

[1] For a good discussion, see Schubring [11].

[2] I am definitely not implying that Aaboe's treatment, discussed in this paper, is such a version.

[3] An allusion to Bell's *Mathematics*: *Queen and servant of science*.

And last but not least, the construction deserves study because it is beautiful mathematics, done with very basic and simple tools. [4]

2 The regular polygons

Figures and configurations that show regularities have always been found interesting. The roses, friezes and tilings of the plane have been widely explored, over the centuries and in countless cultures, by artists and decorators, and studied mathematically, providing a fine example of group theory applied to geometry and crystallography.

The regular polygons, central in geometry, stand out among the important figures that show regularities. The first proposition of Euclid's *Elements* shows how to construct an equilateral triangle. The square also plays an important role in Greek mathematics, in which a major problem was to "square" a figure, that is, to build a square with area equal to the area of that figure. The regular pentagon was important to the Pythagoreans, starting from the sixth century B.C.E., since the pentagram (the regular star polygon of five sides, Figure 1) was the symbol of the Pythagorean brotherhood.

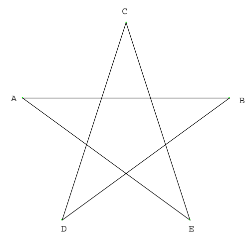

Figure 1: The regular pentagram

The Pythagoreans may have known the fact that the diagonal and the side of the pentagon are incommensurable. This can be seen taking into account that the intersection of the diagonals of the pentagon $ABCDE$ define a new pentagon, $EFGHK$, and so on (Figure 2). If the diagonal and the side of $ABCDE$ are commensurable, the same happens to the sides and diagonals of all the pentagons so obtained, and we arrive at a contradiction. This fact, together with the familiarity of the Pythagoreans with the pentagon, led von Fritz [13] to propose that the existence of incommensurable magnitudes was discovered using the pentagon and its diagonal, and not the square and its diagonal. However, von Fritz's opinion is not the favorite among historians of mathematics.

One of the highlights of the *Elements* of Euclid is the construction, in Book IV, of the regular pentagon. Euclid needs it in book XIII, in which he studies the five regular polyhedra, for the construction of the regular dodecahedron, whose sides are regular pentagons. He uses only the non-graduated ruler and the compass, as happens in all constructions in the *Elements*.

[4] We would like to thank the referees for the excellent suggestions, which helped to improve this paper.

The search for the construction of the regular polygons has been a recurring theme among mathematicians. When they could not find constructions that obey the Euclidean canons, they were able, at least, to find approximate constructions, or which require other resources besides the ruler and compass, as seen, for example, in the construction for the regular heptagon given by Archimedes.

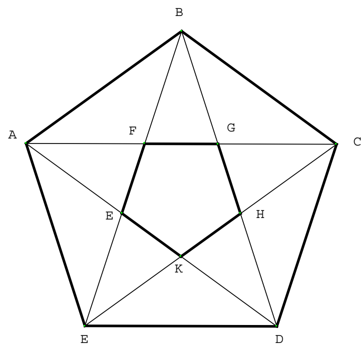

Figure 2: The regular pentagon and its diagonals

A complete answer for which regular polygons can be constructed with ruler and compass was given only in the nineteenth century, by Gauss and Wantzel. The first proved, in 1796, that it is possible to construct with ruler and compass the regular polygon of 17 sides. A little later, in his *Disquisitiones Arithmeticae* [4], he proved that a sufficient condition for the regular n-sided polygon to be constructible with ruler and compass is that n, the number of sides of the polygon, be of the form

$$n = 2^k p_1 p_2 \dots p_s,$$

in which all p's are Fermat primes, that is, primes of the form $p_i = 2^{2^{n_i}} + 1$. Since the Fermat primes known presently are 3, 5, 17, 257 and 65537, it is possible, in principle, to construct the regular polygons with the number of sides equal to these primes or to products thereof. Thus, the regular polygons with less than 300 sides which can be constructed with ruler and compass are the ones with the following number of sides:

3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102, 120, 128, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102, 120, 128, 136, 160, 170, 192, 204, 240, 255, 256, 257, 272 136, 160, 170, 192, 204, 240, 255, 256, 257, 272.

Gauss stated that his condition is also necessary, but this was proved only by Pierre Wantzel, in 1836 ([14]), a French mathematician, in a work that proved the impossibility of solving the problems of angle trisection and of doubling the cube.

The criterion of Gauss does not provide an explicit construction of a regular polygon. The German mathematician Johannes Erchinger gave such a construction, with ruler and compass, for the polygon of 17 sides, in 1825, and Friedrich Julius Richelot proposed, in 1832, a method to construct the polygon of 257 sides, but there are doubts about its validity (see Reich [10]).

3 A well-known construction of the regular pentagon

There are many constructions of the regular pentagon, some of which use only the tools prescribed by Euclid. One, well known and particularly simple, due to Ptolemy, [5] proceeds as follows to inscribe a regular pentagon in the circle of center O and radius OB (Figure 3).

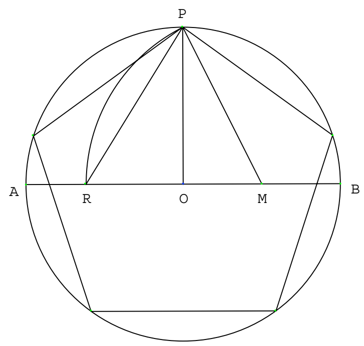

Figure 3: The construction of a regular pentagon inscribed in a circle

Let OP be the perpendicular to the diameter AB, starting from the center O of the circle, and M the midpoint of OB. With center M and radius MP draw the arc of a circle which cuts AB in R. Then RO will be the side of the regular decagon inscribed in the circle of radius OB. Once one knows the side of the regular decagon, it is easy to construct the pentagon, connecting the vertices of the decagon two by two.[6] Note that all the steps in this construction can be made with ruler and compass.

We shall initially justify this construction, following Aaboe ([1], pp. 54-56. See also Aaboe [2]).

Consider, in Figure 4, the triangle OKL, in which $KL = x$ is the side of the regular decagon inscribed in the circle of radius $OK = OL = r$. Thus, the angle $\widehat{LOK}$ measures 36°. Since LOK is an isosceles triangle, its base angles are equal, and so each one measures 72°. With center K and radius $KL = x$, draw a circle, and let T be the point at which it cuts OL. Then, because of the way it was drawn, KLT is an isosceles triangle, and therefore the angle $\widehat{KTL}$ also measures 72°. It follows that the angle $\widehat{TKL}$ measures 36°, and $\widehat{OKT}$ is equal to 36°, and therefore OKT is also an isosceles triangle. Then, $OT = KT = KL = x$.

Since the triangles OKL and KTL are similar, we have

$$\frac{OL}{LK} = \frac{KT}{TL},$$

[5]See Ptolemy, [9], I, 10, On the size of chords in a circle, pp. 14-15.

[6]Euclid shows, in Proposition XIII.10 of the *Elements*, that the sides of the regular dodecagon, pentagon and hexagon inscribed in a circle form a right-angle triangle. This implies that PR will be the side of the regular pentagon.

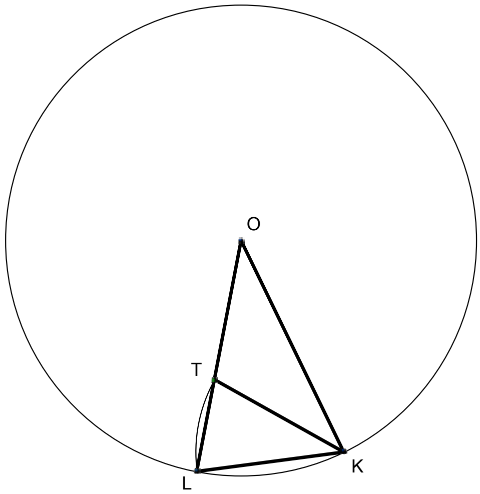

Figure 4: The triangles LKT and KOL are similar

that is,

$$\frac{r}{x} = \frac{x}{r-x} \Longrightarrow x^2 + rx - r^2 = 0.$$

The only positive root of this equation is

$$x = \frac{1}{2}r(\sqrt{5} - 1).$$

Let us return to Figure 3. The triangle POM has a right angle, and therefore

$$PM^2 = r^2 + \left(\frac{r}{2}\right)^2 \Longrightarrow PM = \left(\frac{r}{2}\right)\sqrt{5}.$$

So,

$$OR = RM - OM = PM - OM = \frac{r}{2}\sqrt{5} - \frac{1}{2}r = \frac{1}{2}r\left(\sqrt{5} - 1\right).$$

Thus, the segment OR is the side of the regular decagon. It is now easy to construct the regular pentagon: it is sufficient to join the vertices of the decagon, two by two.

We can make some comments about the treatment presented by Aaboe, which is mathematically correct, but deserves pedagogical and, equally important, historiographic remarks.

Firstly, as far as pedagogy is concerned, Aaboe says simply that it is "more convenient" to find the side of the regular decagon, without explaining why this is so. We will try to show why it is actually more convenient to work with the regular decagon.

Figure 5 shows a pentagon and a regular decagon inscribed in a circle. If we construct the decagon, we can construct the pentagon, joining the vertices of the decagon two by two. It is also true that starting with the pentagon we can construct the decagon. As a matter of fact, if you can construct with ruler and compass the regular polygon of n sides it is possible to construct the regular polygon of $2n$ sides, and conversely.[7]

[7]More generally, if r and s are co-prime natural numbers, and we know how to construct the polygons with respectively r and s sides, it is then possible to construct the polygon with rs sides. Euclid proves this for the case $r = 3$ and $s = 5$ in the last proposition of Book IV (IV.16).

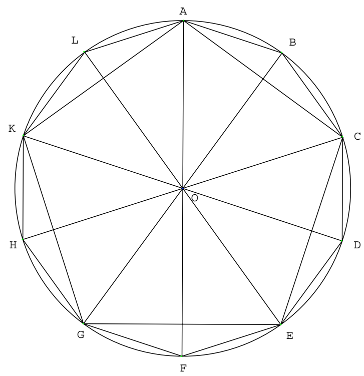

Figure 5: The regular pentagon and decagon inscribed in a circle

Why should one construct the regular decagon and not, directly, the regular pentagon? In Figure 5, the angles $\widehat{DOE}$, $\widehat{EOF}$, $\widehat{GOS}$ $\ldots$, $\widehat{COD}$ are equal, since they are the central angles of the regular decagon $ABCDEFGHKL$. Furthermore, the angle $\widehat{ODE}$ is equal to the angle $\widehat{KDE}$, and so to half the arc $KHGFE$. As this arc is equal to four times the arc DE, we see that the angle $\widehat{ODE}$ is equal to twice the angle $\widehat{DOE}$, that is

$$\widehat{ODE} = 2 \times \widehat{DOE}.$$

Thus, in the isosceles triangle ODE, each base angle is twice the vertex angle. Therefore, to construct the regular decagon inscribed in the circle of radius R, it is enough to construct an isosceles triangle ABC such that its base angles are both twice the vertex angle and then draw in the circle a triangle DEF equiangular with the triangle ABC.

The crucial point is that (See Figure 5) in the isosceles triangle OAB the vertex angle, $\widehat{AOB}$, is equal to half the base angles. As shown by Aaboe, the fact that the triangles OKL and KTL are similar (Figure 4) implies that the radius of the circle, OK, is divided by T in extreme and mean ratio.

Secondly, we have historiographic problems. Aaboe uses the fact that we are dealing with angles that measure 72° and 36°, respectively. However, to measure angles this way is entirely avoided in the *Elements*. In fact, it was unknown in classical Greek mathematics. The only thing necessary, as we explained above, is the relationship between the angles ODE and DOE (Figure 5), which follows from the results contained in the *Elements* about the angles inscribed in a circumference. Euclid never measures lengths, areas, volumes and angles. Since the Greeks did note have the real numbers, they did not measure magnitudes, they dealt directly with these, comparing them. For example, angles are compared with the right angle, polygons are transformed into squares (they are "squared") and the resulting squares are compared. Of course one can add segments, angles, areas and volumes.

Aaboe's presentation is mathematically correct. His mistake is to encase it in Greek dress, giving the reader the impression that the Greeks, in particular Euclid, worked in the manner shown by him, Aaboe.

More serious in Aaboe's presentation is his use of the so-called "Greek geometric algebra",[8] which has been the subject of heated discussion in the last decades. In particular, Sabetai Unguru has denounced, in several papers (see [3] and also [11] for full references), the anachronistic interpretation of Greek and Babylonian mathematics done by the proponents of the "Greek geometric algebra", in the light of modern developments, that is, the widespread algebraization of mathematics. [9]

The anachronism of Aaboe's text is shown particularly in the interpretation of Proposition II.6 of the *Elements* as a way to solve second degree equations, failing to understand its importance in the geometric context of the *Elements*.[10] *Elements* II.5 and II.6 certainly can be interpreted in an algebraic way, totally out of the context of classical Greek mathematics, but then one should not claim to be presenting what Euclid did. [11]

We now present Euclid's construction of the regular pentagon.

4 The construction of the pentagon by Euclid

The construction of the regular pentagon inscribed in a circle is the climax of Book IV of Euclid's *Elements*. It should be stressed that this construction does not use similarity. It is based entirely on equivalence of areas. Once discovered, it can easily be done. We emphasize, moreover, that the construction in Book IV clearly shows the strength of the method of the equivalence of areas, widely used by Euclid, until Book V, exclusive.

Since Euclid, in Book XIII of the *Elements* constructs, with ruler and compass, the five regular polyhedra, he could not avoid constructing, beforehand, the regular pentagon, because the faces of the regular dodecahedron are regular pentagons.

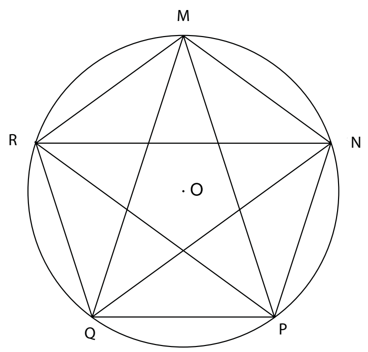

Figure 6: The angle at M is half the angle $\widehat{MQP}$

Euclid, like other classical Greek mathematicians, merely presents his results logically linked. Thus, we give, initially, a brief survey of his construction. We recommend

[8] "Greek geometrical algebra" was created, at the end of the 19th and beginning of the 20th centuries, by Neugebauer, Tannery and Zeuthen (See, for example, [15]). It became very generally accepted because of Heath's influential and well known translations of Greek mathematics. It was continued by van der Waerden, who agreed with this interpretation.

[9] A thoughtful and balanced view of this question can be found in Vitrac ([12], pp. 366-376)

[10] Of course, Ptolemy's original treatment is entirely in accordance with the methods and tools developed in the *Elements* of Euclid.

[11] A broad and deep discussion of this interpretation of Greek mathematics can be found in Fried and Unguru, [3].

returning to this initial description, after reading through the text. The construction itself is done in Propositions 10 and 11 of Book IV.

Euclid starts from the easily seen fact that in the regular pentagon, $MNPQR$ (Figure 6), the isosceles triangle MQP is such that its vertex angle is equal to half of each base angle. If we can inscribe such a triangle in a circle, we can construct the pentagon.

To perform this construction Euclid requires two lines of argumentation. The first, applying II.11 to the radius of the circle, KV, he finds the point Z such that the rectangle on KV and ZV is equal to the square on KZ (See Figure 7). [12] Let T, on the circumference, be such that $VT = KZ$. [13] Then Euclid draws an auxiliary circle, which passes through points K, Z and T [14] and using results about secants and chords in a circle, contained in II.36 and III.37, he shows, finally, that the triangle KVT satisfies the required property, i.e., the angle at its vertex is equal to half of each base angle.

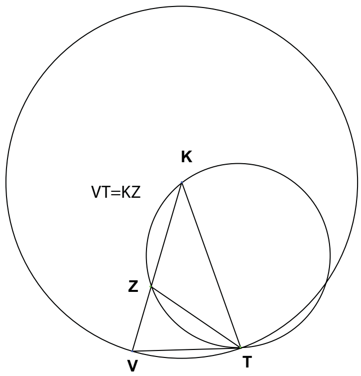

Figure 7: The triangles KVT and ZVT

After this, to construct the regular pentagon inscribed in a given circle, Euclid inscribes in this circle a triangle equiangular with the triangle just constructed, KVT.

We now turn our attention to the actual construction. We state, without proofs, the results from the *Elements* he needs, following Heath's versions ([6] and [7]).

We begin with Proposition II.6, very important in the *Elements*.

Proposition II.6: *If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.*[15]

In other words, the rectangle with sides AD, DB, together with the square on CB, is equal to the square on CD.

Euclid uses also the following result, which is a consruction problem

Proposition II.11:*To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.*

[12]Using the theory of proportions, expounded in Book V, and applied to figures in Book VI, this is equivalent to the division of KV in mean and extreme ratios.

[13]This can be done by proposition I.2.

[14]Euclid shows how to do this IV.5.

[15]We recall that, for Euclid, *straight line* can mean either a segment or the whole straight line.

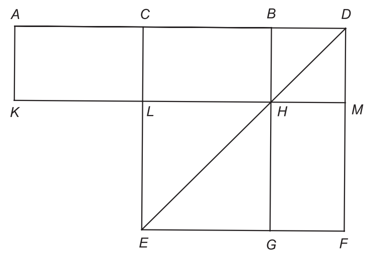

Figure 8: *Elements* II.6

That is, Euclid shows how to find, using only a straightedge and a compass, a point C on AB such that the rectangle of sides AB and CB is equal to the square on AC (Figure 9).

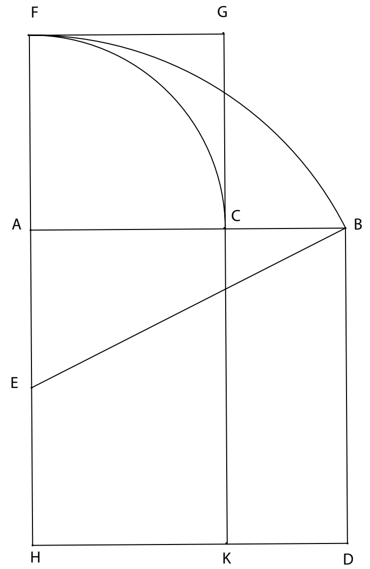

Figure 9: *Elements* II.11

Now we need some results about circles and their chords, from Book III of the *Elements*.

Proposition III.36: *If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.*

That is, the rectangle with sides PR and PS is equal to the square on PT (Figure 10).

Proposition III.37: *If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex*

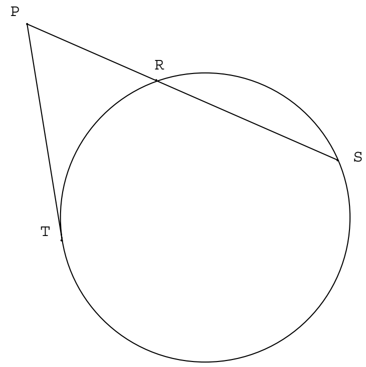

Figure 10: *Elements* III.36

circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.

This proposition is the converse of III.36.

For the last step before reaching the construction of the regular pentagon, we will use two additional results, also without proof:

Proposition III.22: *The opposite angles of quadrilaterals in circles are equal to two right angles.*

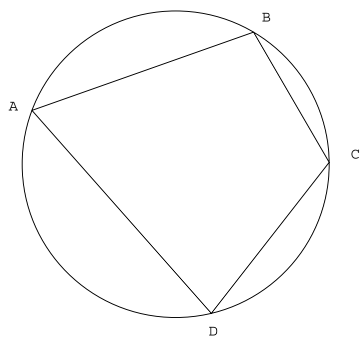

Figure 11: *Elements* III.22

That is, for the quadrilateral $ABCD$ of Figure 11, inscribed in a circle, the angles $\widehat{BAD}$ and $\widehat{BCD}$ are equal, taken together, to two right angles. The same is true of the angles $\widehat{ADC}$ and $\widehat{ABC}$.

Proposition III.31: *In a circle, the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and further, the angle of the greater segment is greater than a right angle, and the angle of the less segment less than a right angle.*

This means that in the circle of Figure 12, the angle $\widehat{BAC}$ is right, the angle $\widehat{ABC}$ in the arc greater than the semicircle is less than a right angle, and the angle $\widehat{ADC}$ in the arc ADC, less than the semicircle, is greater than a right angle.

We need also the following result.

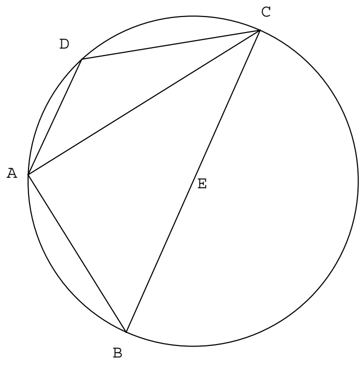

Figure 12: *Elements* III.31

Proposition III.32: *If a straight line touch a circle and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.*

This proposition states that the angle $\widehat{DBF}$ is equal to the angle $\widehat{BAD}$ (Figure 13).

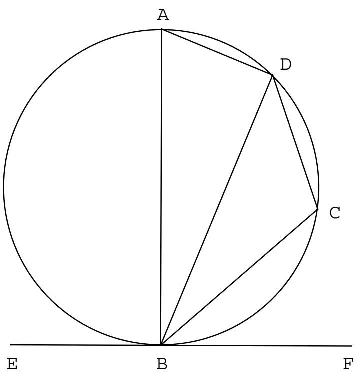

Figure 13: *Elements* III.32

What remains now is just to construct the regular pentagon. The essential step is the following construction problem. Euclid's text can be found in Heath ([7]).

Proposition IV.10: *Construct an isosceles triangle in which each base angle is double the vertex angle.*

Let a line segment AB be given (Figure 14). Using II.11, find the point C such that the rectangle of sides AB, CB is equal to the square on CA.

Draw the circle with center at A and radius AB. From B, draw BD equal to AC. Then, the triangle ABD has the required property, namely, that each of the base angles is equal to twice the vertex angle.

We summarize below, in a symbolic and compact way, the reasoning of Euclid. In what follows, ret (AB, BC) designates the rectangle of sides AB and BC, and quad (AC) represents the square on the segment AC.

The point C is such that ret(AB, BC) = quad(AC).

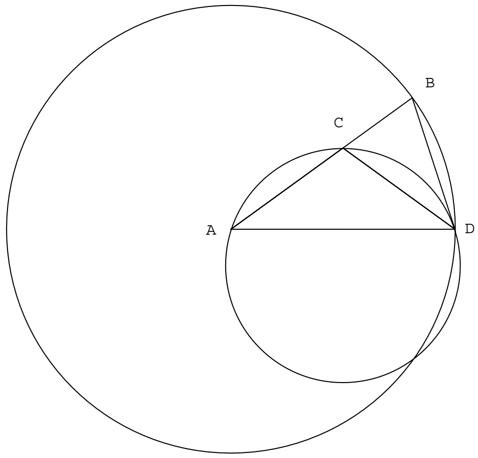

Figure 14: *Elements* IV.10

Draw AD and DC and construct the circle ACD circumscribed to the triangle ACD (This can be done by IV.5).

Since $AC = BD$, by construction, we have

$$\text{ret}(AB,\, BC) = \text{quad}(BD).$$

But then, it follows from III.37, that BD is tangent to the circle ACD.

By III.32, we have

$$\widehat{BDC} = \widehat{DAC} \Longrightarrow \widehat{BDC} + \widehat{CDA} = \widehat{DAC} + \widehat{CDA}.$$

Thus,

$$\widehat{BDA} = \widehat{DAC} + \widehat{CDA}.$$

Consider, in the triangle ACD, the external angle $\widehat{BCD}$. Then

$$\widehat{BCD} = \widehat{DAC} + \widehat{CDA} \Longrightarrow \widehat{BCD} = \widehat{BDA}.$$

Since $\widehat{BDA} = \widehat{DBA}$, we have

$$\widehat{BCD} = \widehat{BDA} = \widehat{DBA} \Longrightarrow \widehat{BCD} = \widehat{DBC} \Longrightarrow DB = DC.$$

Since $DB = CA$, it follows that $CA = CD$, and therefore $\widehat{CAD} = \widehat{CDA}$.

Thus,

$$\widehat{CAD} + \widehat{CDA} = 2\widehat{CAD} \Longrightarrow \widehat{BCD} = 2 \times \widehat{CAD},$$

and therefore

$$\widehat{BCD} = \widehat{BDA} = \widehat{DBA} = 2 \times \widehat{CAD} = 2 \times \widehat{BAD}.$$

Thus, in the triangle ABD, each base angle is twice the vertex angle. $\square$

Before the final construction, we need

Proposition IV.2: *In a given circle to inscribe a triangle equiangular with a given triangle.*

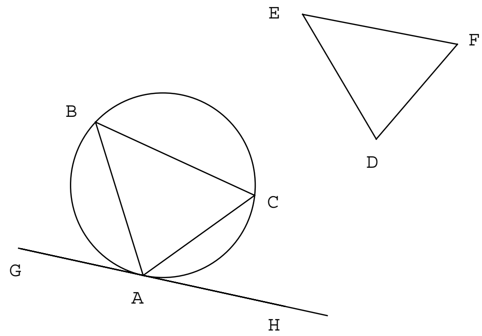

Figure 15: *Elements* IV.2

Let the circle and the triangle DEF be given (Figure 15). Let GH be the tangent to the circle, passing through the point A. Draw the angle $\widehat{HAC}$ equal to the angle $\widehat{DEF}$ and the angle $\widehat{GAB}$ equal to the angle $\widehat{DFE}$. Draw BC. Then, by III.32, the angle $\widehat{HAC}$ is equal to the angle $\widehat{ABC}$ and therefore the angle $\widehat{ABC}$ is equal to the angle $\widehat{DEF}$. Similarly, we can show that the angles $\widehat{ACB}$ and $\widehat{DFE}$ are equal. Therefore, the angle $\widehat{BAC}$ will be equal to the angle $\widehat{EDF}$.

Therefore, in the given circle there has been inscribed a triangle, with angles respectively equal to the angles of the given triangle.

□

We can finally present the result we set out to prove.

Proposition IV.11: *In a given circle to inscribe an equilateral and equiangular pentagon.*

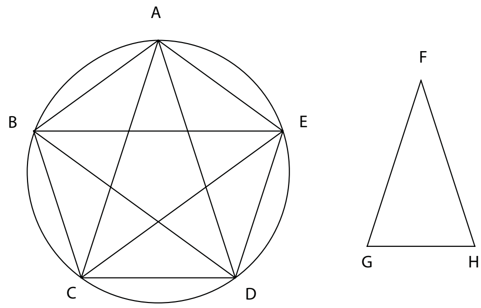

Figure 16: *Elements* IV.11

Construct the triangle FGH in which each base angle is twice the vertex angle. Inscribe the triangle ACD in the circle, with the angle $\widehat{CAD}$ equal to the angle at F and the angles $\widehat{ACD}$ and $\widehat{CDA}$ equal, respectively, to the angles at G and H (Figure 16).

Bisect the angles $\widehat{ACD}$ and $\widehat{ADC}$ by EC and DB, respectively. Draw the straight line segments AB, BC, DE and EA.

Since each one of the angles $\widehat{ACD}$ and $\widehat{CDA}$ is twice the angle $\widehat{CAD}$, and have been bisected by the lines CE and DB, respectively, it follows that the angles $\widehat{DAC}$, $\widehat{ACE}$,

$\widehat{ECD}$, $\widehat{CDB}$ and $\widehat{BDA}$ are equal to one another. But then the arcs AB, BC, CD, DE, EA are equal, and so the line segments AB, BC, CD, DE, EA are also equal.

Therefore the pentagon $ABCDE$ is equilateral.

Now, since the arc AB is equal to the arc DE, if we add the arc BCD to both, it follows that the arc $ABCD$ is equal to the arc $EDCB$. From this we see that the angle $\widehat{BAE}$ is equal to the angle $\widehat{AED}$.

For the same reason, each of the angles $\widehat{ABC}$, $\widehat{BCD}$, $\widehat{CDE}$ is also equal to each of the angles $\widehat{BAE}$ and $\widehat{AED}$.

Therefore the pentagon is equiangular, and so it is a regular pentagon.

□

Obviously, as noted by Hartshorne ([5], pp. 45-51), Euclid's construction can be modified, to become quicker and more efficient. It is enough to apply IV.10 directly to the circle in which we want to inscribe the regular pentagon.

REFERENCES

1. AABOE, Asger. *Episodes from the early history of mathematics*. Washington, DC: Mathematical Association of America, 1964.
2. AABOE, Asger. *Episódios da história antiga da Matemática*. Rio de Janeiro: Sociedade Brasileira de Matemática, 1997.
3. FRIED, Michael N. and UNGURU, Sabetai. *Apollonius of Perga's Conica*: *text, context, subtext*. Mnemosyne. Supplementum. 222. Leiden: Brill. xii, 499 p. 2001.
4. GAUSS, Carl Friedrich. *Disquisitiones Arithmeticae*. Werke, vol 1, König. Ges. der Wiss. zu Göttingen, 1870. English translation by Arthur Clarke, S.J. New Haven, Conn: Yale University Press, 1966.
5. HARTSHORNE, Robin. *Geometry*: *Euclid and beyond*. New York: Springer, 2000.
6. HEATH, T.L. *The thirteen books of the Elements of Euclid*, vol. 1 (Books I-II). New York: Dover, 1956.
7. HEATH, T.L. *The thirteen books of the Elements of Euclid*, vol. 2 (Books III-IX). New York: Dover, 1956.
8. HEATH, T.L. *The thirteen books of the Elements of Euclid*, vol. 3 (Books X-XIII). New York: Dover, 1956.
9. PTOLEMY, Claudius. *Almagest I - IV*, translated by R. Catesby Taliaferro. In Hutchins, Robert Maynard (ed.) *Ptolemy, Copernicus, Kepler*. Great books of the western world, vol. 16. Encyclopaedia Britannica, 1848, Chigago, Ill.
10. REICH, Karin. "Die Entdeckung und frühe Rezeption der Konstruierbarkeit des regelmaßigen 17-Ecks und dessen geometrische Konstruktion durch Johannes Erchinger (1825)". In: *Mathesis, Festschrift zum siebzigsten Geburtstag von Matthias Schramm*. Hrsg. von Rudiger Thiele, Berlin: Diepholz 2000, s.101-118.
11. SCHUBRING, Gert. "The debate on a geometric algebra and methodological implications". In *Proceedings of HPM - Mexico*, Ciudad de Mexico, 2008.

12. VITRAC, Bernard. *Les Éléments*, vol 1. Introduction générale, Livres I IV. Paris: Presses Universitaires de France, 1990.

13. von FRITZ, Kurt. "The discovery of incommensurability by Hipparcus of Metapontum". *Annals of mathematics*, second series, vol 46, No 2 (April 1945), pp. 242-264.

14. WANTZEL, Pierre Laurent. "Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas". *Journal de mathématiques pures et appliquées*, vol. 1, 1836, pp. 366-372.

15. ZEUTHEN, H.G. *Die Lehre der Kegelschnitte im Altertum.* Kopenhagen: Host, 1886.